

Khovanov's Heisenberg category, moments in free probability, and shifted symmetric functions

Henry Kvinge, Anthony M. Licata*, and Stuart Mitchell

Abstract

We establish an isomorphism between the center $\text{End}_{\mathcal{H}'}(\mathbb{1})$ of the Heisenberg category defined by Khovanov in [13] and the algebra Λ^* of shifted symmetric functions defined by Okounkov-Olshanski in [18]. We give a graphical description of the shifted power and Schur bases of Λ^* as elements of $\text{End}_{\mathcal{H}'}(\mathbb{1})$, and describe the curl generators of $\text{End}_{\mathcal{H}'}(\mathbb{1})$ in the language of shifted symmetric functions. This latter description makes use of the transition and co-transition measures of Kerov [10] and the noncommutative probability spaces of Biane [2].

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1 Introduction

In [13], Khovanov introduces a graphical calculus of oriented planar diagrams and uses it to define a linear monoidal category \mathcal{H}' , which he proposes as a categorification of the Heisenberg algebra. We denote by $\text{End}_{\mathcal{H}'}(\mathbb{1})$ the endomorphism algebra of the monoidal unit in \mathcal{H}' . The commutative algebra $\text{End}_{\mathcal{H}'}(\mathbb{1})$ is, by definition, the algebra of closed oriented planar diagrams modulo the relations of the Khovanov graphical calculus. In his study of morphism spaces of \mathcal{H}' , Khovanov introduces two sets of generators for $\text{End}_{\mathcal{H}'}(\mathbb{1})$: the clockwise curls $\{c_k\}_{k \geq 0}$ and the counterclockwise curls $\{\tilde{c}_k\}_{k \geq 2}$. He then establishes algebra isomorphisms

$$\text{End}_{\mathcal{H}'}(\mathbb{1}) \cong \mathbb{C}[c_0, c_1, c_2, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \dots],$$

and describes a recursion for expressing the clockwise and counterclockwise curls in terms of each other. He then relates \mathcal{H}' to representation theory by defining a sequence of monoidal functors $f_k^{\mathcal{H}'}$ from \mathcal{H}' to bimodule categories for symmetric groups. A consequence of the existence of these functors is the existence of surjective algebra homomorphisms,

$$f_n^{\mathcal{H}'} : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow Z(\mathbb{C}[S_n]),$$

from $\text{End}_{\mathcal{H}'}(\mathbb{1})$ to the center of the group algebra of each symmetric group. Based in part on this, Khovanov suggests that there should be a close connection between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and the asymptotic representation theory of symmetric groups. Furthermore, one might hope that $\text{End}_{\mathcal{H}'}(\mathbb{1})$ in fact gives a diagrammatic description of some algebra of pre-existing combinatorial interest.

The main goal of the current paper is to make precise the connection between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and both the asymptotic representation theory of symmetric groups and algebraic combinatorics. We do this by establishing an isomorphism between

$$\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*,$$

where Λ^* is the *shifted symmetric functions* of Okounkov-Olshanski [18]. (See Theorem 5.3.) The algebra of shifted symmetric functions Λ^* is a deformation of the algebra of symmetric functions. As is the case for $\text{End}_{\mathcal{H}'}(\mathbb{1})$, there are surjective algebra homomorphisms

$$f_n^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}[S_n]),$$

to the center of the group algebra of each symmetric group. The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ is canonical, in that it intertwines the homomorphisms $f_n^{\mathcal{H}'}$ and $f_n^{\Lambda^*}$.

The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ allows us to give a graphical description of several important bases of Λ^* . For example, the shifted power sum denoted $p_\lambda^\#$ in [18] appears in $\text{End}_{\mathcal{H}'}(\mathbb{1})$ as the closure of a permutation of cycle type λ . The shifted Schur function s_λ^* appears as the closure of a Young symmetrizer of type λ . (See Theorem 5.4).

In the other direction, it is also reasonable to ask for a description of the image of Khovanov’s curl generators c_k and \tilde{c}_k as elements of Λ^* . It turns out that the right language for such a description is that of noncommutative probability theory. In [10], Kerov introduces, for each partition λ , a pair of finitely supported probability measures on \mathbb{R} ; these probability measures are known as the *transition* and *co-transition* measures, or sometimes as growth and decay. In work of Biane [2], these probability measures appear as the compactly-supported measures associated to self-adjoint operators on a noncommutative probability space, and as a result they are basic objects of interest at the intersection of representation theory and noncommutative probability theory. In particular, the *moments* and *Boolean cumulants* of the transition and co-transition measures may be regarded as elements of Λ^* . In Theorem 5.5, we show that the isomorphism φ takes Khovanov’s curl generators c_k and \tilde{c}_k to scalar multiples of the k th moments of Kerov’s transition and co-transition measures. In fact, the close relationship between the transition and co-transition measures themselves yields two independent descriptions of the image of the curl generator c_k : it is equal to a scalar multiple of both the k th moment of the co-transition measure and the $(k + 2)$ th Boolean cumulant of the transition measure. The observation that the Boolean cumulants of the transition measure are equal to the moments of the co-transition measure seems to be new, and is closely connected to the adjointness of induction and restriction functors between representation categories of symmetric groups. A dictionary between several of the bases of $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and Λ^* is given in Table 1 below.

The existence of a relationship between \mathcal{H}' and free probability – and indeed, much of this paper – was anticipated by Khovanov in [13]. The relationship between generators of $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and the noncommutative probability spaces of [2] may be seen as a further manifestation of the “planar structure” of free probability; the many connections between noncommutative probability and other mathematical subjects with planar structure are emphasized in the work of Guionnet, Jones and Shlyakhtenko [6].

In addition to the center of \mathcal{H}' , another algebra of interest in the study of \mathcal{H}' is its trace (or zeroth Hochschild homology). The trace of \mathcal{H}' is an infinite-dimensional noncommutative algebra, which may be defined diagrammatically as the algebra of diagrams on an annulus; the trace acts naturally on $\text{End}_{\mathcal{H}'}(\mathbb{1})$ by gluing annular diagrams around planar ones. In [4], the trace of \mathcal{H}' is shown to be isomorphic to the $W_{1+\infty}$ algebra of conformal field theory. An action of $W_{1+\infty}$ on Λ^* appears to be well known in the vertex algebra community, and such an action is constructed explicitly in the work of Lascoux-Thibon [14]. Thus the isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \rightarrow \Lambda^*$ of Theorem 5.3, together with the main result of [4], gives a purely planar realization – via Khovanov’s graphical calculus – of Lascoux-Thibon’s construction.

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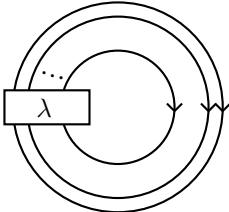
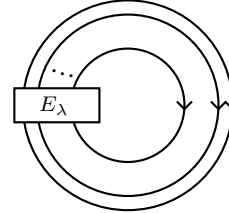
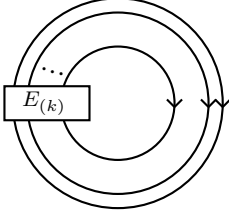
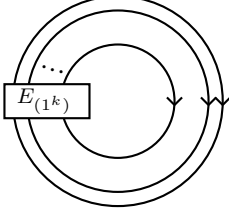


Λ^*	diagram in $\text{End}_{\mathcal{H}'}(\mathbf{1})$
$p_\lambda^\#$	
s_λ^*	$\frac{1}{\dim L^\lambda}$ 
h_k^*	
e_k^*	
\hat{m}_k	
$\hat{b}_{k+2} = p_1^\# \tilde{m}_k$	

Table 1: A dictionary between Λ^* and diagrams in $\text{End}_{\mathcal{H}'}(\mathbf{1})$.

2 The symmetric group and its normalized character theory

We begin by establishing notation related to partitions and Young diagrams. Let \mathcal{P}_n be the set of partitions of n and

$$\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}_n.$$

For this section let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathcal{P}_n$ and $\mu = (\mu_1, \dots, \mu_t) \in \mathcal{P}_k$ with $n \geq k$. We assume that $\lambda_1 \geq \dots \geq \lambda_r > 0$ and $\mu_1 \geq \dots \geq \mu_t > 0$. When $i > r$ (respectively $i > t$) we then understand $\lambda_i = 0$ (resp. $\mu_i = 0$). We use the following notation throughout:

- $n = \lambda_1 + \lambda_2 + \dots + \lambda_r =: |\lambda|$.
- $\lambda \cup \mu$ is the partition formed from the union of the parts of λ and μ .
- $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$. When this is the case, we write λ/μ for the associated skew diagram.
- $\phi_{k,n} : \mathcal{P}_k \hookrightarrow \mathcal{P}_n$ is the function defined by $\phi_{k,n}(\mu) = \mu \cup 1^{n-k} \in \mathcal{P}_n$.

Example 2.1. If $\mu = (3, 2, 1, 1, 1) \in \mathcal{P}_8$ then $\phi_{8,10}(\mu) = (3, 2, 1, 1, 1, 1, 1) \in \mathcal{P}_{10}$.

We freely identify $\mu \in \mathcal{P}$ with its corresponding Young diagram, which we draw using Russian notation (see Example 2.2). If \square is a cell in the i th row and j th column of μ then the *content* of \square is defined as

$$\text{cont}(\square) := j - i.$$

We say that a cell $\square \notin \mu$ is i -addable with respect to μ if it has content i and adding it to μ gives a Young diagram. We say that a cell $\square \in \mu$ is i -removable with respect to μ if it has content i and removing it from μ gives a Young diagram. We call two sequences a_1, \dots, a_d and b_1, \dots, b_{d-1} *interlacing* when

$$a_1 < b_1 < a_2 < \dots < a_{d-1} < b_{d-1} < a_d.$$

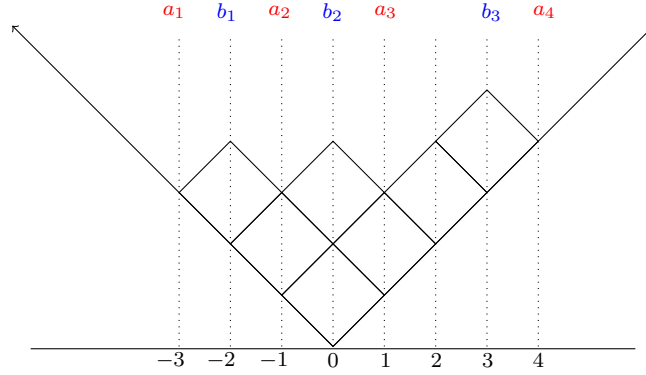
The *center* of this pair of sequences is defined as the quantity $(a_1 + \dots + a_d) - (b_1 + \dots + b_{d-1})$. Each Young diagram μ uniquely defines two integer valued interlacing sequences a_1, \dots, a_d and b_1, \dots, b_{d-1} where:

- a_1, \dots, a_d is the ordered list of all a_j such that there exists an a_j -addable cell with respect to μ .
- b_1, \dots, b_{d-1} is the ordered list of all b_j such that there exists a b_j -removable cell with respect to μ .

From this description it is clear that a_1, \dots, a_d and b_1, \dots, b_{d-1} are interlacing.

Example 2.2. Let $\mu = (4, 2, 1)$. Then μ yields the interlacing sequences

$$\textcolor{red}{-3} < \textcolor{red}{-1} < \textcolor{red}{1} < \textcolor{red}{4} \quad \text{and} \quad -2 < 0 < 3.$$



Proposition 2.3. [11] If a_1, \dots, a_d and b_1, \dots, b_{d-1} are the pair of interlacing sequences associated to a Young diagram then their center is 0. Conversely, any pair of integer valued interlacing sequences with center 0 are associated to a Young diagram.

When $\mu \subseteq \lambda$ and $\lambda/\mu = \square$, then we write $\mu \nearrow \lambda$. In other words, $\mu \nearrow \lambda$ whenever we can obtain λ from μ by adding a single cell. If a_1, \dots, a_d and b_1, \dots, b_{d-1} are the interlacing sequences associated to μ , then we denote by $\mu^{(i)}$ the Young diagram that we get by adding a cell of content a_i , so that

$$\text{cont}(\mu^{(i)}/\mu) = a_i.$$

Similarly, we denote by $\mu_{(i)}$ the Young diagram that we get by removing a cell of content b_i from μ , so that

$$\text{cont}(\mu/\mu_{(i)}) = b_i.$$

Note that $\mu_{(i)} \nearrow \mu$, while $\mu \nearrow \mu^{(i)}$.

Example 2.4. If $\mu = (4, 2, 1)$ as in Example 2.2, we have

$$\begin{array}{ll} \textcolor{red}{\mu^{(1)}} &= (4, 2, 1, 1) \\ \textcolor{red}{\mu^{(2)}} &= (4, 2, 2) \\ \textcolor{red}{\mu^{(3)}} &= (4, 3, 1) \\ \textcolor{red}{\mu^{(4)}} &= (5, 2, 1) \end{array} \quad \text{and} \quad \begin{array}{ll} \mu_{(1)} &= (4, 2) \\ \mu_{(2)} &= (4, 1, 1) \\ \mu_{(3)} &= (3, 2, 1). \end{array} \quad (1)$$

Let S_n be the symmetric group. S_n is generated by Coxeter generators s_1, \dots, s_{n-1} where s_i is the adjacent transposition $(i, i+1)$. We identify $\mathbb{C}[S_0] \cong \mathbb{C}$. If $g \in S_n$ has cycle type $\lambda \vdash n$, then we write $\text{sh}(g) := \lambda$. For $k \leq n$, there is an embedding $S_k \hookrightarrow S_n$ called the *standard embedding* which sends S_k to the subgroup generated by s_1, \dots, s_{k-1} , which stabilizes $\{k+1, \dots, n\}$ pointwise.

We extend this embedding by linearity to get an embedding of group algebras which we denote by $\iota_{k,n} : \mathbb{C}[S_k] \hookrightarrow \mathbb{C}[S_n]$. We write 1_k for the identity element in $\mathbb{C}[S_k]$ so that $\iota_{k,n}(1_k) = 1_n$. We write $w_{0,n}$ for the longest element of S_n .

For $\lambda \vdash n$, let L^λ be the simple $\mathbb{C}[S_n]$ -module corresponding to λ , E_λ its associated Young idempotent, and $\chi^\lambda : \mathbb{C}[S_n] \rightarrow \mathbb{C}$ its associated character. Abusing notation, we write $\chi^\lambda(\mu)$ for $\chi^\lambda(g)$ when $\text{sh}(g) = \mu$ (this notation is well-defined since χ^λ is a class function). The *normalized character* $\tilde{\chi}^\lambda : \bigoplus_{k \leq n} \mathbb{C}[S_k] \rightarrow \mathbb{C}$ associated to λ is defined so that for $x \in \mathbb{C}[S_k]$,

$$\tilde{\chi}^\lambda(x) := \frac{\chi^\lambda(\iota_{k,n}(x))}{\dim L^\lambda} = \frac{\chi^\lambda(\iota_{k,n}(x))}{\chi^\lambda(1_n)}. \quad (2)$$

Let $\mu = (\mu_1, \dots, \mu_t) \vdash k \leq n$ and set $\pi_\mu = 1_k$ if $\mu = (1^k)$ and otherwise

$$\begin{aligned} \pi_\mu &= \left(s_{k-1} \dots s_{k-\mu_t+1} \right) \dots \left(s_{\mu_1+\mu_2-1} \dots s_{\mu_1+1} \right) \dots \left(s_{\mu_1-1} \dots s_2 s_1 \right) \\ &= (k, k-1, \dots, k-\mu_t+1)(\mu_1 + \mu_2, \dots, \mu_1 + 1) \dots (\mu_1, \dots, 2, 1) \in S_k. \end{aligned}$$

We define

$$\sigma_{\mu,n} := w_{0,n}^{-1}(\iota_{k,n}(\pi_\mu))w_{0,n} \in S_n.$$

Observe that $\sigma_{\mu,n}$ has cycle type $\phi_{k,n}(\mu)$ and fixes $1, 2, \dots, n-k$ pointwise.

Example 2.5. Let $\mu = (3, 2) \vdash 5$, then

$$\pi_\mu = (s_4)(s_2 s_1) = (5, 4)(3, 2, 1)$$

and we see that $\text{sh}(\pi_\mu) = \mu$. For $n = 8$,

$$\sigma_{\mu,8} = s_4 s_6 s_7 = (4, 5)(6, 7, 8),$$

while for $n = 10$,

$$\sigma_{\mu,10} = (6, 7)(8, 9, 10).$$

The elements

$$\{1_n, \sigma_{(2),n}, \sigma_{(3),n}, \dots, \sigma_{(n),n}\} = \{1_n, s_{n-1}, s_{n-2}s_{n-1}, \dots, s_1 s_2 \dots s_{n-1}\}$$

are the minimal length left coset representatives of S_{n-1} in S_n . We extend this observation in the following lemma.

Lemma 2.6. For $k < n$, the elements of the set

$$\{\sigma_{(i_n),n} \sigma_{(i_{n-1}),n-1} \dots \sigma_{(i_{k+1}),k+1} \mid 1 \leq i_j \leq j\}$$

are the minimal length left coset representatives of S_k in S_n . We denote this set by \mathcal{LC}_k^n .

We note that $|\mathcal{LC}_k^n| = (n \downarrow n - k)$, where the *falling factorial power* is defined as

$$(x \downarrow k) = \begin{cases} x(x-1)\dots(x-k+1), & \text{if } k = 1, 2, \dots \\ 1, & \text{if } k = 0. \end{cases}$$

Example 2.7. We have

$$\mathcal{LC}_3^4 = \{1_4, \textcolor{red}{s}_3, \textcolor{red}{s}_2\textcolor{red}{s}_3, \textcolor{red}{s}_1\textcolor{red}{s}_2\textcolor{red}{s}_3\},$$

$$\mathcal{LC}_2^3 = \{1_3, \textcolor{blue}{s}_2, \textcolor{blue}{s}_1\textcolor{blue}{s}_2\},$$

and

$$\begin{aligned} \mathcal{LC}_2^4 = \{ & 1_4, \textcolor{red}{s}_3, \textcolor{red}{s}_2\textcolor{red}{s}_3, \textcolor{red}{s}_1\textcolor{red}{s}_2\textcolor{red}{s}_3, \\ & \textcolor{blue}{s}_2, \textcolor{red}{s}_3\textcolor{blue}{s}_2, \textcolor{red}{s}_2\textcolor{red}{s}_3\textcolor{blue}{s}_2, \textcolor{red}{s}_1\textcolor{red}{s}_2\textcolor{red}{s}_3\textcolor{blue}{s}_2, \\ & \textcolor{blue}{s}_1\textcolor{red}{s}_2, \textcolor{red}{s}_3\textcolor{blue}{s}_1\textcolor{red}{s}_2, \textcolor{red}{s}_2\textcolor{red}{s}_3\textcolor{blue}{s}_1\textcolor{red}{s}_2, \textcolor{red}{s}_1\textcolor{red}{s}_2\textcolor{red}{s}_3\textcolor{blue}{s}_1\textcolor{red}{s}_2\}. \end{aligned}$$

2.1 The center of $\mathbb{C}[S_n]$

For $\mu \vdash k \leq n$, set

$$C_{\mu,n} := \sum_{\substack{g \in S_n, \\ \text{sh}(g) = \phi_{k,n}(\mu)}} g.$$

The elements $\{C_{\mu,n}\}_{\mu \vdash n}$ are a basis for the center of the symmetric group algebra, $Z(\mathbb{C}[S_n])$. We write $z_{\mu,n}$ for the size of the centralizer of an element in S_n with cycle type $\phi_{k,n}(\mu)$. Note that when $\mu \vdash n$, then $z_{\mu,n} = z_\mu$.

Definition 2.8. For $\mu = (\mu_1, \dots, \mu_t) \vdash k \leq n$, set

$$A_{\mu,n} := \sum_{g \in \mathcal{LC}_{n-k}^n} g \sigma_{\mu,n} g^{-1}. \quad (3)$$

We call $A_{\mu,n}$ the *normalized conjugacy class sum* associated to μ in $\mathbb{C}[S_n]$.

Alternatively, $A_{\mu,n}$ may be written as

$$A_{\mu,n} = \sum (i_1, \dots, i_{\mu_1}) \dots (i_{k-\mu_t+1}, \dots, i_k) \quad (4)$$

where this sum is taken over all distinct k -tuples (i_1, \dots, i_k) of elements from $\{1, 2, \dots, n\}$. From (4) an easy counting argument shows that

$$A_{\mu,n} = \frac{z_{\mu,n}}{(n-k)!} C_{\mu,n}. \quad (5)$$

It follows from (5) that $A_{\mu,n} \in Z(\mathbb{C}[S_n])$.

Example 2.9. Let $k \leq n$. When $\mu = (k) \vdash k$, then $z_{(k),n} = k(n-k)!$ so that

$$A_{(k),n} = kC_{(k),n}.$$

The elements $A_{\mu,n}$ are important in the study of the asymptotic character theory of symmetric groups [12]. They also appear in connection with the algebra of partial permutations [8]. If $\mu \vdash k \leq n$ and $\lambda \vdash n$ then

$$\tilde{\chi}^\lambda(A_{\mu,n}) = (n \downarrow k) \frac{\chi^\lambda(\mu)}{\dim L^\lambda}. \quad (6)$$

The following is well-known.

Proposition 2.10. When restricted to $Z(\mathbb{C}[S_n])$, the normalized character $\tilde{\chi}^\lambda$ is an algebra homomorphism from $Z(\mathbb{C}[S_n])$ to \mathbb{C} .

$Z(\mathbb{C}[S_n])$ is also generated by symmetric polynomials in the Jucys-Murphy elements $\{J_i\}_{1 \leq i \leq n} \subseteq \mathbb{C}[S_n]$, where

$$J_1 = 0, \quad \text{and} \quad J_k = (1, k) + (2, k) + \cdots + (k-1, k), \quad 2 \leq k \leq n.$$

We can also write

$$J_k = \sum_{i=1}^{k-1} s_i \cdots s_{k-2} s_{k-1} s_{k-2} \cdots s_i. \quad (7)$$

2.2 The transition measure and co-transition measure

In this section we recall the notion of transition and co-transition measures, also known as growth and decay, respectively. Assume that $\lambda \vdash n$ and let a_1, \dots, a_d and b_1, \dots, b_{d-1} be the interlacing sequences associated to λ . Recall that $\lambda^{(1)}, \dots, \lambda^{(d)}$ are the partitions of $n+1$ such that $\text{cont}(\lambda^{(i)}/\lambda) = a_i$, while $\lambda_{(1)}, \dots, \lambda_{(d-1)}$ are the partitions of $n-1$ such that $\text{cont}(\lambda/\lambda_{(i)}) = b_i$.

For $1 \leq i \leq d$, the *transition probabilities* for λ are defined as

$$\hat{q}_\lambda(\lambda^{(i)}) := \frac{\dim(L^{\lambda^{(i)}})}{(n+1) \dim(L^\lambda)}.$$

The *transition measure* $\hat{\omega}_\lambda$ is then the probability measure on \mathbb{R} defined by

$$\hat{\omega}_\lambda := \sum_{i=1}^d \hat{q}_\lambda(\lambda^{(i)}) \delta_{a_i} \quad (8)$$

where δ_{a_i} is the Dirac delta measure with support on $a_i \in \mathbb{R}$. Dually, for $1 \leq i \leq d-1$ the *co-transition probabilities* of λ are

$$\check{q}_\lambda(\lambda_{(i)}) := \frac{\dim(L^{\lambda_{(i)}})}{\dim(L^\lambda)}$$

and the *co-transition measure* $\check{\omega}_\lambda$ is

$$\check{\omega}_\lambda := \sum_{i=1}^{d-1} \check{q}_\lambda(\lambda_{(i)}) \delta_{b_i}. \quad (9)$$

These probability measures were first investigated by Kerov ([10], [11]). They are fundamental tools in the study of the asymptotic representation theory of symmetric groups and in the connection between asymptotic representation theory and free probability.

The k th moment associated to the transition measure $\hat{\omega}_\lambda$ is given by

$$\hat{m}_k(\lambda) = \sum_{i=1}^d a_i^k \hat{q}_\lambda(\lambda^{(i)})$$

while the k th moment associated to the co-transition measure $\check{\omega}_\lambda$ is given by

$$\check{m}_k(\lambda) = \sum_{i=1}^{d-1} b_i^k \check{q}_\lambda(\lambda_{(i)}).$$

We write the moment generating series for the transition measure (resp. co-transition measure) as

$$\widehat{\mathcal{M}}_\lambda(z) := \sum_{k=0}^{\infty} \hat{m}_k(\lambda) z^{-k-1} \quad \text{and} \quad \widetilde{\mathcal{M}}_\lambda(z) := z - \sum_{k=0}^{\infty} |\lambda| \check{m}_k(\lambda) z^{-k-1}.$$

Note that we scale all coefficients of $\widetilde{\mathcal{M}}_\lambda(z)$ by $|\lambda|$ with the exception of the coefficient on z .

Lemma 2.11. For $\lambda \in \mathcal{P}$

$$\widehat{\mathcal{M}}_\lambda(z) = (\widetilde{\mathcal{M}}_\lambda(z))^{-1}. \quad (10)$$

Proof. This follows directly from equation (2.3) and Lemma 5.1 in [11]. \square

The *boolean cumulants* $\{\hat{b}_k(\lambda)\}_{k \geq 1}$ associated to $\hat{\omega}_\lambda$ can be defined as the coefficients on the multiplicative inverse of $\widehat{\mathcal{M}}_\lambda(z)$,

$$\hat{\mathcal{B}}_\lambda(z) = z - \sum_{k=-1}^{\infty} \hat{b}_{k+2}(\lambda) z^{-k-1} = (\widehat{\mathcal{M}}_\lambda(z))^{-1}. \quad (11)$$

With Lemma 2.11 this definition immediately gives us the following fact.

Proposition 2.12. Let $\lambda \in \mathcal{P}$ and $k \geq 0$, then $\hat{b}_1(\lambda) = 0$ and

$$\hat{b}_{k+2}(\lambda) = |\lambda| \check{m}_k(\lambda). \quad (12)$$

Remark 2.13. The equality (11) can be rewritten as

$$\sum_{i=1}^k \hat{m}_{k-i}(\lambda) \hat{b}_i(\lambda) = \hat{m}_k(\lambda). \quad (13)$$

For general information about the relationship between moments, Boolean cumulants, and other families of cumulants see [1].

There is a more algebraic approach to the transition measure due to Biane [2]. Let

$$\text{pr}_{n-1} : \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_n]$$

be the projection map defined on S_n by

$$\text{pr}_{n-1}(g) = \begin{cases} g & \text{if } g(n) = n \\ 0 & \text{otherwise.} \end{cases}$$

In the context of probability theory, pr_{n-1} is sometimes known as the *conditional expectation*.

Proposition 2.14. For $\lambda \vdash n$,

$$\hat{m}_k(\lambda) = \tilde{\chi}^\lambda[\text{pr}_n(J_{n+1}^k)] \quad (14)$$

and

$$\hat{b}_{k+2}(\lambda) = |\lambda| \tilde{m}_k(\lambda) = \tilde{\chi}^\lambda \left(\sum_{i=1}^n s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i \right). \quad (15)$$

Proof. The statement of (14) appears in [3] Section 4. A detailed proof is given in Theorem 9.23 of [7]. To get (15) note that since characters are class functions,

$$\tilde{\chi}^\lambda \left(\sum_i^n s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i \right) = |\lambda| \tilde{\chi}^\lambda(J_n^k).$$

As J_n eigenspaces, L^λ decomposes as

$$L^\lambda \cong \bigoplus_{i=1}^{d-1} L^{\lambda(i)}$$

with $L^{\lambda(i)}$ corresponding to eigenvalue b_i [20]. Hence,

$$|\lambda| \tilde{\chi}^\lambda(J_n^k) = |\lambda| \sum_{i=1}^{d-1} \frac{\dim(\lambda(i)) b_i^k}{\dim(\lambda)} = |\lambda| \tilde{m}_k(\lambda) = \hat{b}_{k+2}(\lambda).$$

□

Proposition 2.14 is related to the fact that we are working in a noncommutative probability space (that is, a von Neumann algebra equipped with a normal faithful trace). In our case the algebra is $\text{End}(L^\lambda) \otimes M_{n+1}(\mathbb{C})$ and $\hat{\omega}_\lambda$ then arises from the distribution of a self-adjoint element in this algebra (see Proposition 3.3 in [2]).

3 Symmetric functions and shifted symmetric functions

In order to define the algebra of shifted symmetric functions, we first recall the classical symmetric functions. Let Λ_n be the algebra of symmetric polynomials over \mathbb{C} in x_1, \dots, x_n . This algebra is graded by polynomial degree. Recall that for $n \geq 0$ there is a homomorphism

$$\Lambda_{n+1} \rightarrow \Lambda_n \quad (16)$$

given by setting $x_{n+1} = 0$ in Λ_{n+1} . One can define the algebra of symmetric functions as the projective limit $\Lambda = \varprojlim \Lambda_n$ taken in the category of graded algebras. We recall three collections of algebraically independent generators of Λ :

- elementary symmetric functions e_1, e_2, e_3, \dots ,
- complete homogeneous symmetric functions h_1, h_2, h_3, \dots ,
- power sum symmetric functions p_1, p_2, p_3, \dots

For $\{f_k\}_{k \geq 1}$ equal to any of these three sets of generators and $\lambda = (\lambda_1, \dots, \lambda_r)$ we write $f_\lambda := f_{\lambda_1} \dots f_{\lambda_r}$. We denote the basis of Schur functions by $\{s_\lambda\}_{\lambda \in \mathcal{P}}$. We refer the reader to [16] and [19] for background on Λ .

Let Λ_n^* be the algebra of polynomials over \mathbb{C} in x_1, \dots, x_n , which become symmetric in the new variables $x'_i = x_i - i$. This algebra is filtered by polynomial degree. In analogy to Λ_{n+1} , setting $x_{n+1} = 0$ in Λ_{n+1}^* gives a homomorphism

$$\Lambda_{n+1}^* \rightarrow \Lambda_n^* \quad (17)$$

which respects the filtration. Using (17), set

$$\Lambda^* := \varprojlim \Lambda_n^*,$$

where this limit is taken in the category of filtered algebras. Λ^* is called the *algebra of shifted symmetric functions*.

Because Λ^* is filtered, we can consider the associated graded algebra $\text{gr}(\Lambda^*)$.

Proposition 3.1. [18, Prop. 1.5] $\text{gr}(\Lambda^*)$ is canonically isomorphic to Λ .

Remark 3.2. It is noted in Remark 1.7 of [18] that we may also view Λ^* as a deformation of Λ . Let $\Lambda_n^*(\theta)$ be the algebra of polynomials in x_1, \dots, x_n which are symmetric in the new variables $x'_i = x_i + c - i\theta$ for $1 \leq i \leq n$ and where $c \in \mathbb{C}$. Define $\Lambda^*(\theta) = \varprojlim \Lambda_n^*(\theta)$. Then $\Lambda^*(0) = \Lambda$ and $\Lambda^*(1) = \Lambda^*$. In fact for all $\theta \neq 0$, $\Lambda^*(\theta) \cong \Lambda^*$.

3.1 Bases of Λ^*

In [18] Okounkov and Olshanski introduced a remarkable basis for Λ^* called the shifted Schur functions. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ (note that here we allow components of a partition to be zero). The *shifted Schur polynomial in n variables, indexed by λ* is the ratio of two $n \times n$ determinants,

$$s_\lambda^*(x_1, \dots, x_n) = \frac{\det[(x_i + n - i \downarrow \lambda_j + n - j)]}{\det[(x_i + n - i \downarrow n - j)]}, \quad (18)$$

where $1 \leq i, j \leq n$. This polynomial belongs to Λ_n^* . It is shown in [18] that

$$s_\lambda^*(x_1, \dots, x_n, 0) = s_\lambda^*(x_1, \dots, x_n). \quad (19)$$

This implies that for fixed λ , letting $n \rightarrow \infty$ gives a well-defined element s_λ^* of Λ^* . The elements $\{s_\lambda^*\}_{\lambda \in \mathcal{P}} \in \Lambda^*$ are called the *shifted Schur functions* and form a basis for Λ^* . There is a linear map $\Lambda^* \rightarrow \text{gr}(\Lambda^*) \cong \Lambda$ which sends $f \in \Lambda^*$ to its top homogeneous component which is an element of Λ . Under this map

$$s_\lambda^* \mapsto s_\lambda$$

or alternatively,

$$s_\lambda^* = s_\lambda + \text{l.o.t.} \quad (20)$$

where l.o.t. means lower order terms in polynomial degree.

In analogy to the classical case, the *elementary shifted functions* can be defined as $e_k^* := s_{(1^k)}^*$, while the *complete shifted functions* can be defined as $h_k^* := s_{(k)}^*$. More explicitly:

$$e_k^*(x_1, x_2, \dots) = \sum_{1 \leq i_1 < \dots < i_k < \infty} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \dots x_{i_k}$$

and

$$h_k^*(x_1, x_2, \dots) = \sum_{1 \leq i_1 \leq \dots \leq i_k < \infty} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \dots x_{i_k}.$$

Let F be the linear isomorphism $F : \Lambda \rightarrow \Lambda^*$ which sends $s_\lambda \mapsto s_\lambda^*$. Define the element $p_\lambda^\# \in \Lambda^*$ to then be

$$p_\lambda^\# := F(p_\lambda), \quad (21)$$

where p_λ is the power sum symmetric function. The elements $p_\lambda^\#$ are one of several shifted analogues of the power sums. For $\lambda \vdash n$, the transition coefficients between the power-sum and Schur bases are given by the character tables of the symmetric group (see [19]):

$$p_\lambda = \sum_{\mu \vdash n} \chi^\mu(\lambda) s_\mu.$$

It follows directly from definition (21) that

$$p_\lambda^\# = \sum_{\mu \vdash n} \chi^\mu(\lambda) s_\mu^*. \quad (22)$$

Note also that by (20) and (22),

$$p_\lambda^\# = p_\lambda + \text{l.o.t.} \quad (23)$$

Since the power symmetric functions p_1, p_2, \dots are algebraically independent and generate Λ , it follows from (23) that $p_1^\#, p_2^\#, \dots$ are algebraically independent and generate Λ^* . Similarly, since $\{p_\lambda\}_{\lambda \in \mathcal{P}}$ is a basis for Λ , $\{p_\lambda^\#\}_{\lambda \in \mathcal{P}}$ is a basis for Λ^* . For more properties of the basis $\{p_\lambda^\#\}$ see [9].

Remark 3.3. Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. While it is true that in Λ , $p_{\lambda_1} \dots p_{\lambda_r} = p_\lambda$, in general

$$p_{\lambda_1}^\# \dots p_{\lambda_r}^\# \neq p_\lambda^\#.$$

However, by (23)

$$p_{\lambda_1}^\# \dots p_{\lambda_r}^\# = p_\lambda^\# + \text{l.o.t.}$$

3.2 Λ^* as functions on \mathcal{P}

Let $\text{Fun}(\mathcal{P}, \mathbb{C})$ be the algebra of functions from \mathcal{P} to \mathbb{C} with pointwise multiplication. Viewing $\mu = (\mu_1, \dots, \mu_t) \vdash k$ as the sequence $(\mu_1, \dots, \mu_t, 0, 0, \dots)$, we can evaluate $f \in \Lambda^*$ on μ by setting

$$f(\mu) = f(\mu_1, \dots, \mu_t, 0, 0, \dots). \quad (24)$$

Since $(\mu_1, \dots, \mu_t, 0, 0, \dots)$ has only a finite number of nonzero values, it is clear that (24) is well-defined. In fact f is uniquely defined by its values on \mathcal{P} . Thus Λ^* may be realized as a subalgebra of $\text{Fun}(\mathcal{P}, \mathbb{C})$. This fact is used repeatedly en route to establishing many of the fundamental results about shifted symmetric functions in [12] and [18].

For $\lambda \vdash n$ and α a cell in the Young diagram corresponding to λ let $h(\alpha)$ be the hook length of α . Then set $H(\lambda)$ as the product of all hooklengths in λ ,

$$H(\lambda) := \prod_{\alpha \in \lambda} h(\alpha).$$

The following is known as the ‘‘Characterization Theorem’’ of [17].

Theorem 3.4. For $\mu \vdash k$, s_μ^* is the unique element of Λ^* such that $\deg(s_\mu^*) \leq k$ and

$$s_\mu^*(\lambda) = \delta_{\mu\lambda} H(\mu)$$

for all $\lambda \in \mathcal{P}$ such that $|\lambda| \leq |\mu|$.

This theorem along with (22) then give the following proposition.

Proposition 3.5. [18] For $\mu \vdash k$, $\lambda \vdash n$,

$$p_\mu^\#(\lambda) = \begin{cases} \frac{(n|k)}{\dim L^\lambda} \chi^\lambda(\mu) & k \leq n \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

Remark 3.6. We will later use the fact that $p_1^\# = x_1 + x_2 + \cdots = p_1$, so that $p_1^\#(\lambda) = |\lambda|$ for all $\lambda \in \mathcal{P}$.

In Section 2.2 we introduced the moments $\{\hat{m}_k(\lambda)\}$ (resp. $\{\check{m}_k(\lambda)\}$) of the transition measure (resp. co-transition measure) associated to a partition λ and the corresponding Boolean cumulants $\{\hat{b}_k(\lambda)\}$. We can interpret all of these as elements of $\text{Fun}(\mathcal{P}, \mathbb{C})$ via

$$\lambda \xrightarrow{\hat{m}_k} \hat{m}_k(\lambda), \quad \lambda \xrightarrow{\check{m}_k} \check{m}_k(\lambda), \quad \text{and} \quad \lambda \xrightarrow{\hat{b}_k} \hat{b}_k(\lambda).$$

We omit the partition argument from \hat{m}_k , \check{m}_k , and \hat{b}_k in this context to emphasize that we are considering them as elements of $\text{Fun}(\mathcal{P}, \mathbb{C})$.

Proposition 3.7. [15, Theorem 6.4] As elements of $\text{Fun}(\mathcal{P}, \mathbb{C})$, \hat{m}_k and \hat{b}_k belong to Λ^* .

Remark 3.8. In [15] Section 5, Lassalle shows that with the appropriate alphabet A_λ (which is specific to each partition λ),

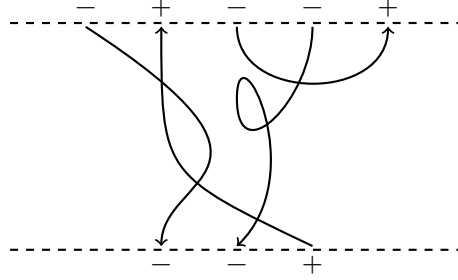
$$\hat{m}_k(\lambda) = h_k(A_\lambda) \quad \text{and} \quad \hat{b}_k(\lambda) = (-1)^{k-1} e_k(A_\lambda). \quad (26)$$

4 The Heisenberg category \mathcal{H}'

In [13], Khovanov defined an additive \mathbb{C} -linear monoidal category \mathcal{H}' which we will call the *Heisenberg category*. The objects in \mathcal{H}' are generated by two objects Q_+ and Q_- . Following the notation of [13], we denote $Q_{\epsilon_1} \otimes \cdots \otimes Q_{\epsilon_m}$ by Q_ϵ where $\epsilon = \epsilon_1 \dots \epsilon_m$ is a finite sequence of pluses and minuses. The unit object, $\mathbb{1}$, corresponds to the empty sequence Q_\emptyset .

The collection of morphisms $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$, for two sequences ϵ and ϵ' is the \mathbb{C} -vector space spanned by planar diagrams modulo some local relations. The diagrams are oriented compact 1-manifolds embedded in the strip $\mathbb{R} \times [0, 1]$, modulo rel boundary isotopies. The endpoints of the 1-manifolds are located at $\{1, \dots, m\} \times \{0\}$ and $\{1, \dots, n\} \times \{1\}$, where m and n are the lengths of ϵ and ϵ' , respectively. Further, the orientation of the 1-manifold at the endpoints must match the signs in the sequences ϵ and ϵ' . Triple intersections are not allowed.

Example 4.1. The diagram



is a morphism from Q_{--+} to Q_{-++} .

The composition of two morphisms is achieved by stacking diagrams. The local relations for diagrams are:

$$\begin{array}{c} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} - \begin{array}{c} \text{cup} \\ \text{cap} \end{array} \end{array} \quad (27)$$

$$\begin{array}{c} \text{circle} \\ \text{cup} \end{array} = 1 \quad \begin{array}{c} \text{cup} \\ \text{cap} \end{array} = 0 \quad (28)$$

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array} = \begin{array}{c} \nearrow \\ \nwarrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array} \quad (29)$$

The relations (27) and (28) are motivated by the Heisenberg relation $pq = qp + 1$, where p and q are the two generators of the Heisenberg algebra, while the relations (29) are motivated by the symmetric group relations.

It is convenient to denote a right curl by a dot on a strand, and a sequence of d right curls by a dot with a d next to it:

$$\begin{array}{c} \uparrow \\ \bullet \\ | \end{array} := \begin{array}{c} \uparrow \\ \curvearrowright \end{array}, \quad d \begin{array}{c} \uparrow \\ \bullet \\ | \end{array} := \begin{array}{c} \uparrow \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ | \end{array} \left. \vphantom{\begin{array}{c} \uparrow \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ | \end{array}} \right\} d \text{ dots} .$$

A right curl can be moved across intersection points, according to the following “dot-sliding relations” [13]:

$$\begin{array}{c} \nearrow \bullet \nwarrow \\ \swarrow \searrow \end{array} = \begin{array}{c} \nearrow \nwarrow \\ \swarrow \bullet \searrow \end{array} + \begin{array}{c} \uparrow \\ | \end{array} \begin{array}{c} \uparrow \\ | \end{array},$$

$$\begin{array}{c} \nwarrow \bullet \swarrow \\ \nearrow \searrow \end{array} = \begin{array}{c} \nwarrow \swarrow \\ \nearrow \bullet \searrow \end{array} + \begin{array}{c} \uparrow \\ | \end{array} \begin{array}{c} \uparrow \\ | \end{array}.$$

This observation easily generalizes to

$$\begin{array}{c} \nearrow \bullet^k \nwarrow \\ \swarrow \searrow \end{array} = \begin{array}{c} \nearrow \nwarrow \\ \swarrow \bullet^k \searrow \end{array} + \sum_{i=0}^{k-1} \begin{array}{c} \uparrow \\ | \end{array} \begin{array}{c} \uparrow \\ \bullet^{k-1-i} \\ | \end{array}, \quad (30)$$

$$\begin{array}{c} \nwarrow \bullet^k \swarrow \\ \nearrow \searrow \end{array} = \begin{array}{c} \nwarrow \swarrow \\ \nearrow \bullet^k \searrow \end{array} + \sum_{i=0}^{k-1} \begin{array}{c} \uparrow \\ | \end{array} \begin{array}{c} \uparrow \\ \bullet^{k-1-i} \\ | \end{array}. \quad (31)$$

Another consequence of relations (27)-(29) are the “bubble moves” [13]:

$$\begin{array}{c} \bullet^k \curvearrowright \\ | \end{array} = \begin{array}{c} \bullet^k \curvearrowright \\ | \end{array} + (k+1) \begin{array}{c} \uparrow \\ \bullet^k \\ | \end{array} - \sum_{i=0}^{k-2} (k-i-1) \begin{array}{c} \uparrow \\ \bullet^{k-i-2} \\ | \end{array} \begin{array}{c} \bullet^i \curvearrowright \end{array}, \quad (32)$$

$$\begin{array}{c} \uparrow \\ \bullet^k \curvearrowright \end{array} = \begin{array}{c} \bullet^k \curvearrowright \\ | \end{array} - \sum_{i=0}^{k-2} (k-i-1) \begin{array}{c} \bullet^i \curvearrowright \\ | \end{array} \begin{array}{c} \uparrow \\ \bullet^{k-i-2} \\ | \end{array}. \quad (33)$$

Note that relations (29) imply that there is a homomorphism $\mathcal{T}_n : \mathbb{C}[S_n] \rightarrow \text{End}_{\mathcal{H}'}(Q_{+^n})$ which sends

$$s_k \xrightarrow{\mathcal{T}_n} \begin{array}{c} \uparrow \quad \dots \quad \uparrow \quad \text{X} \quad \uparrow \quad \dots \quad \uparrow \\ \underbrace{\hspace{1.5cm}}_{k-1 \text{ strands}} \quad \underbrace{\hspace{1.5cm}}_{n-k-1 \text{ strands}} \end{array} .$$

Diagrammatically, for $x \in \mathbb{C}[S_n]$ we set

$$\mathcal{T}_n(x) =: \begin{array}{c} \uparrow \uparrow \dots \uparrow \\ \boxed{x} \\ \downarrow \downarrow \dots \downarrow \\ \underbrace{\hspace{1.5cm}}_{n \text{ strands}} \end{array} .$$

The appearance of the group algebra $\mathbb{C}[S_n]$ as endomorphisms in \mathcal{H}' is responsible for the connection between \mathcal{H}' and the representation theory of symmetric groups.

4.1 The endomorphism algebra $\text{End}_{\mathcal{H}'}(\mathbb{1})$

Let $\text{End}_{\mathcal{H}'}(\mathbb{1})$ denote the center of \mathcal{H}' , that is, the algebra of endomorphisms of the monoidal unit object $\mathbb{1}$. Diagrammatically, the algebra $\text{End}_{\mathcal{H}'}(\mathbb{1})$ is the commutative \mathbb{C} -algebra spanned by all closed diagrams, with multiplication given by juxtaposition of diagrams. The algebra structure of $\text{End}_{\mathcal{H}'}(\mathbb{1})$ was determined by Khovanov in [13]. Let $\mathbb{C}[c_0, c_1, c_2, \dots]$ be the polynomial algebra in countably many indeterminants $\{c_i\}_{i \geq 0}$.

Theorem 4.2. [13, Prop. 3] The map $\psi_0 : \mathbb{C}[c_0, c_1, \dots] \rightarrow \text{End}_{\mathcal{H}'}(\mathbb{1})$ which sends

$$c_k \xrightarrow{\psi_0} \begin{array}{c} k \\ \bullet \\ \curvearrowright \end{array} \quad (34)$$

is an algebra isomorphism.

Henceforth we will freely identify c_k with its image in $\text{End}_{\mathcal{H}'}(\mathbb{1})$. Another natural set of diagrams to consider are the counterclockwise-oriented circles with k right-twist curls on them. Set

$$\tilde{c}_k := \begin{array}{c} k \\ \bullet \\ \curvearrowleft \end{array} .$$

It follows from the relations in (28) that $\tilde{c}_0 = 1$ and $\tilde{c}_1 = 0$.

Lemma 4.3. [13, Prop. 2] For $k > 0$,

$$\tilde{c}_{k+1} = \sum_{i=0}^{k-1} \tilde{c}_i c_{k-1-i}. \quad (35)$$

The final class of elements in $\text{End}_{\mathcal{H}'}(\mathbb{1})$ we consider are those arising from the closure of permutations (that is, closures of morphisms in the image of \mathcal{T}_n). We define

$$\text{Box } k \text{ with } k \text{ strands} = \text{Box } k \text{ with strands in a circular permutation}$$

For $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, let

$$\text{Box } \lambda := \text{Box } \lambda_1 \cdots \text{Box } \lambda_r \quad (36)$$

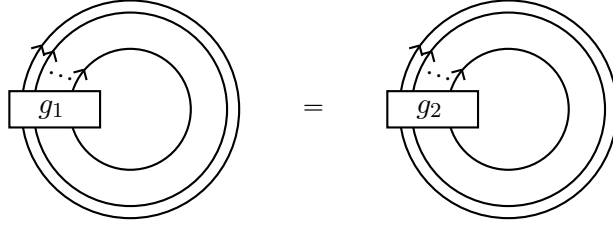
then we define

$$\alpha_\lambda := \text{Box } \lambda \text{ with strands in a circular permutation}$$

with $\alpha_k := \alpha_{(k)}$.

Lemma 4.4 below shows that we could replace the permutation in (36) by the image under \mathcal{T}_n of any $g \in S_n$ such that $\text{sh}(g) = \lambda$. We choose (36) because it will be convenient for later calculations.

Lemma 4.4. Suppose that $g_1, g_2 \in S_n$ are conjugate, so that $\text{sh}(g_1) = \text{sh}(g_2)$. Then



Proof. This is an easy diagrammatic argument which uses the fact that $g_1 = hg_2h^{-1}$ for some $h \in S_n$. Replacing g_1 by hg_2h^{-1} , we slide h around the diagram to cancel it with h^{-1} . \square

4.2 Diagrams as bimodule homomorphisms

In order to establish an isomorphism between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and Λ^* , we will make use of some representations of the monoidal category \mathcal{H}' constructed in [13].

To describe these representations, we start by setting some notation for $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$ -bimodules. All inclusions are assumed to be the standard ones $\iota_{k,n} : S_k \rightarrow S_n$ introduced in Section 2. Suppose that $k_1, k_2 \leq n$. We write:

- (n) for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_n])$ -bimodule.
- $(n)_{k_2}$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_{k_2}])$ -bimodule.
- $_{k_1}(n)$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_n])$ -bimodule.
- $_{k_1}(n)_{k_2}$ for $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$ -bimodule.

Let \mathcal{S}' be the category whose objects are compositions of induction and restriction functors of symmetric groups. We write

$$\text{Ind}_n^{n+1} := \text{Ind}_{S_n}^{S_{n+1}} \quad \text{and} \quad \text{Res}_n^{n+1} := \text{Res}_{S_n}^{S_{n+1}}.$$

Since induction from S_n to S_{n+1} is given by tensoring on the left by $(n+1)_n$ and restriction from S_{n+1} to S_n is given by tensoring on the left by ${}_n(n+1)$, the objects in \mathcal{S}' can be reinterpreted as $(\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])$ -bimodules for $k_1, k_2 \geq 0$.

Example 4.5. One object in \mathcal{S}' is the composition

$$\text{Res}_4^5 \circ \text{Ind}_4^5 \circ \text{Ind}_3^4 \circ \text{Res}_3^4. \quad (37)$$

In the language of bimodules, this is the $(\mathbb{C}[S_4], \mathbb{C}[S_4])$ -bimodule

$$_4(5)_4(4)_3(4).$$

The morphisms in \mathcal{S}' are certain natural transformations of these compositions (or, equivalently, certain bimodule homomorphisms). Like \mathcal{H}' , morphisms in \mathcal{S}' can be presented diagrammatically as oriented compact 1-manifolds embedded in $\mathbb{R} \times [0, 1]$. Unlike \mathcal{H}' , in \mathcal{S}' we label the regions of the strip $\mathbb{R} \times [0, 1]$ by non-negative integers, so that if there is an upwards oriented line separating two regions and the right region is labeled by n , then the left region must be labeled by $n + 1$. The diagram

$$n + 1 \quad \uparrow \quad n$$

denotes the identity endomorphism of the induction functor Ind_n^{n+1} or alternatively the identity endomorphism of the bimodule $(n + 1)_n$.

If there is a downward oriented line separating two regions and the right is labeled by $n + 1$ then the left must be labeled by n . The diagram

$$n \quad \downarrow \quad n + 1$$

denotes the identity endomorphism of the restriction functor Res_n^{n+1} or alternatively the identity endomorphism of the bimodule ${}_n(n + 1)$.

The bimodule maps associated to the four U-turns are:

$$\begin{array}{c} \text{---} n \text{---} \\ \curvearrowright \\ n + 1 \end{array}, \quad (n + 1)_n(n + 1) \rightarrow (n + 1), \quad g \otimes h \mapsto gh, \quad g, h \in S_{n+1}, \quad (38)$$

$$\begin{array}{c} n \\ \curvearrowleft \\ n + 1 \end{array}, \quad (n) \rightarrow {}_n(n + 1)_n, \quad g \mapsto g, \quad g \in S_n, \quad (39)$$

$$\begin{array}{c} \text{---} n + 1 \text{---} \\ \curvearrowleft \\ n \end{array}, \quad {}_n(n + 1)_n \rightarrow (n), \quad g \mapsto \text{pr}_n(g) = \begin{cases} g & g(n + 1) = n + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (40)$$

$$\begin{array}{c} n \\ \curvearrowright \\ n + 1 \end{array}, \quad (n + 1) \rightarrow (n + 1)_n(n + 1), \quad (41)$$

where the last map is determined by the condition that

$$1_{n+1} \mapsto \sum_{i=1}^{n+1} s_i s_{i+1} \cdots s_n \otimes s_n \cdots s_{i+1} s_i = \sum_{g \in \mathcal{LC}_n^{n+1}} g \otimes g^{-1}.$$

Finally, the upward crossing is the bimodule map

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \quad n, \quad (n+2)_n \rightarrow (n+2)_n, \quad g \mapsto gs_{n+1}, \quad g \in S_{n+2}. \quad (42)$$

Any diagram that has a region labeled with a negative number is set to $\mathbf{0}$. It is shown in [13] that all diagrams are compatible with isotopy.

Remark 4.6. Closed diagrams in \mathcal{S}' with outside region labeled by n correspond to $(\mathbb{C}[S_n], \mathbb{C}[S_n])$ -bimodule endomorphisms of (n) . The algebra of such bimodule endomorphisms is isomorphic to $Z(\mathbb{C}[S_n])$ via the map which sends $f \in \text{End}_{(\mathbb{C}[S_n], \mathbb{C}[S_n])}(\mathbb{C}[S_n])$ to $f(1_n)$. Thus closed diagrams in \mathcal{S}' may be regarded as elements of the center of the group algebra.

Khovanov shows that the diagrams in \mathcal{S}' satisfy the defining relations for morphisms in \mathcal{H}' . As a result, given an endomorphism of \mathcal{H}' , after labeling the far right region by a non-negative integer, one obtains a well-defined bimodule homomorphism in \mathcal{S}' . An additional relation that can be calculated directly from the definitions of oriented cups and caps is the following:

$$\begin{array}{c} \circlearrowleft \\ n \end{array} \quad n+1 = n+1. \quad (43)$$

In other words, the endomorphism $c_0 \in \text{End}_{\mathcal{H}'}(1)$ becomes the scalar $n+1$ in $Z(\mathbb{C}[S_{n+1}])$.

\mathcal{S}' is the direct sum of categories

$$\mathcal{S}' = \bigoplus_{k=0}^{\infty} \mathcal{S}'_k,$$

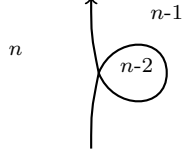
where \mathcal{S}'_k contains all objects such that induction or restriction starts at k (i.e. the rightmost region of the diagram is labeled by k). There are functors $f_k^{\mathcal{H}'} : \mathcal{H}' \rightarrow \mathcal{S}'_k$ such that the object $\epsilon_1 \epsilon_2 \dots \epsilon_n$ is taken to a composition of induction and restriction functors with $+$ sent to Ind_i^{i+1} and $-$ sent to Res_{i-1}^i where i in each case is determined by the requirement that induction/restriction begin from S_k . $f_k^{\mathcal{H}'}$ takes a diagram from \mathcal{H}' to \mathcal{S}'_k by labeling regions so that the rightmost region is labeled with a k and then interpreting the diagram as an element of \mathcal{S}'_k .

Example 4.7. $f_5^{\mathcal{H}'} : \mathcal{H}' \rightarrow \mathcal{S}'_5$ takes

$$\begin{aligned} (+ + - + -) &\xrightarrow{f_5^{\mathcal{H}'}} \text{Ind}_5^6 \circ \text{Ind}_4^5 \circ \text{Res}_4^5 \circ \text{Ind}_4^5 \circ \text{Res}_4^5, \\ (- + +) &\xrightarrow{f_5^{\mathcal{H}'}} \text{Res}_6^7 \text{Ind}_6^7 \text{Ind}_5^6. \end{aligned}$$

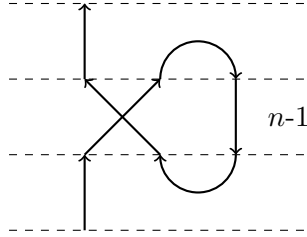
In the remainder of this section we calculate the image of a number of important diagrams in \mathcal{H}' under the functors $f_k^{\mathcal{H}'}$.

Lemma 4.8. [13, Section 4] The diagram



is the endomorphism of $(n)_{n-1}$ which is right multiplication by J_n .

Proof. The right twist curl can be written as the composition of a cup, a crossing, and a cap.



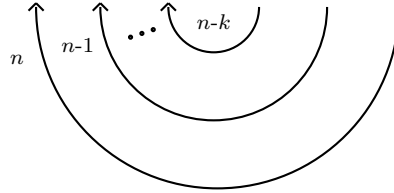
Applying the endomorphism to 1_n gives

$$\begin{aligned} 1_n &\mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} \otimes s_{n-2} \cdots s_i \mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-2} \cdots s_i \\ &\mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_i = J_n \end{aligned}$$

where the equality holds by (7). □

Lemma 4.9. Let $k \leq n$:

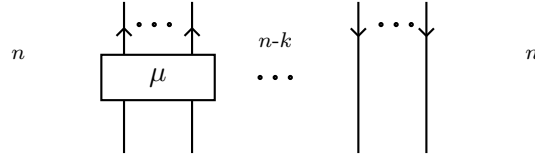
1. The diagram



corresponds to the bimodule homomorphism $(n) \rightarrow (n)_{n-k}(n)$ which sends

$$1_n \mapsto \sum_{g \in \mathcal{LC}_{n-k}^n} g \otimes g^{-1}.$$

2. Let $\mu \vdash k$ and $x_1, x_2 \in (n)$. The diagram



corresponds to the bimodule homomorphism $(n)_{n-k}(n) \rightarrow (n)_{n-k}(n)$ which sends

$$x_1 \otimes x_2 \mapsto x_1 \sigma_{\mu, n} \otimes x_2.$$

Proof. These follow from direct calculation using the definitions of cups, caps, and crossings. \square

Lemma 4.10. As elements of $Z(\mathbb{C}[S_n])$:

1. $f_n^{\mathcal{H}'}(c_k) = \sum_{i=1}^n s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i,$
2. $f_n^{\mathcal{H}'}(\tilde{c}_k) = \text{pr}_n(J_{n+1}^k).$
3. $f_n^{\mathcal{H}'}(\alpha_\mu) = \begin{cases} A_{\mu, n} & \text{if } |\mu| \leq n \\ 0 & \text{otherwise.} \end{cases}$

Proof. (1)-(2) are found in [13] Section 4 and can be computed from the definitions of cups and caps and Lemma 4.8. (3) can be computed by composing the maps in Lemma 4.9 with a sequence of $|\mu|$ nested clockwise oriented caps from (38). When $|\mu| > n$ then $f_n^{\mathcal{H}'}(\alpha_\mu)$ will have its inner region labeled by $n - |\mu| < 0$ and will therefore be 0. \square

5 The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$

In this section we establish the algebra isomorphism $\text{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*$. The proof is somewhat analogous to Ivanov and Kerov's proof of a related isomorphism connecting shifted symmetric functions to the representation theory of symmetric groups (see Theorem 9.1 in [8]).

In [13] Section 4, Khovanov defines a grading on $\text{End}_{\mathcal{H}'}(\mathbb{1})$ by setting

$$\deg(c_0) := 0, \quad \text{and} \quad \deg(c_k) = k + 1, \quad \text{for } k \geq 1. \quad (44)$$

We will consider the increasing filtration induced by this grading. A relationship between the elements $\{c_k\}_{k \geq 0}$ and $\{\alpha_k\}_{k \geq 1}$ is then given in terms of this filtration as follows.

Proposition 5.1. For any $k \geq 1$,

$$\alpha_k = c_{k-1} + \text{l.o.t.}$$

Proof. This follows from repeated application of the dot sliding moves (30)-(31) and bubble sliding move (32). Notice that with each application of these moves, we get a single term from the same filtered part plus additional terms of lower degree. \square

Since the elements c_0, c_1, \dots are algebraically independent generators of $\text{End}_{\mathcal{H}'}(\mathbb{1})$, we immediately obtain the following.

Corollary 5.2. The elements $\alpha_1, \alpha_2, \dots$ are algebraically independent generators of $\text{End}_{\mathcal{H}'}(\mathbb{1})$.

For any $\lambda \vdash n$, composing $f_n^{\mathcal{H}'}$ with the normalized character $\tilde{\chi}^\lambda$ gives a map

$$(\tilde{\chi}^\lambda \circ f_n^{\mathcal{H}'}) : \text{End}_{\mathcal{H}'}(\mathbb{1}) \rightarrow \mathbb{C}$$

and allows us to define a homomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \rightarrow \text{Fun}(\mathcal{P}, \mathbb{C})$. Specifically, for $x \in \text{End}_{\mathcal{H}'}(\mathbb{1})$, we write

$$[\varphi(x)](\lambda) := (\tilde{\chi}^\lambda \circ f_n^{\mathcal{H}'}) (x).$$

Combining Lemma 4.10.3 with (6) implies that for $\mu \vdash k$

$$[\varphi(\alpha_\mu)](\lambda) = \begin{cases} \frac{(n!k)}{\dim L^\lambda} \chi^\lambda(\mu) & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases} \quad (45)$$

Theorem 5.3. The map φ induces an algebra isomorphism $\text{End}_{\mathcal{H}'}(\mathbb{1}) \rightarrow \Lambda^* \subseteq \text{Fun}(\mathcal{P}, \mathbb{C})$ with

$$\alpha_\mu \xrightarrow{\varphi} p_\mu^\#.$$

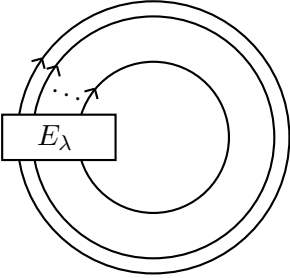
Proof. Let $\lambda \vdash n$. φ is an algebra homomorphism because $f_n^{\mathcal{H}'}$ is a homomorphism from $\text{End}_{\mathcal{H}'}(\mathbb{1})$ to $Z(\mathbb{C}[S_n])$ and $\tilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[S_n])$. By Proposition 3.5 and (45), α_μ maps to $p_\mu^\#$. Since the $\{p_k^\#\}_{k \geq 1}$ (respectively $\{\alpha_k\}_{k \geq 1}$) are algebraically independent generators of Λ^* (resp. $\text{End}_{\mathcal{H}'}(\mathbb{1})$), φ must be an isomorphism. \square

Note that Theorem 5.3 along with Lemma 4.4 imply that when $\mu \vdash n$,

$$\begin{array}{c} \text{Diagram with } C_{\mu,n} \end{array} = \frac{n!}{z_{\mu,n}} \begin{array}{c} \text{Diagram with } \lambda \end{array} \xrightarrow{\varphi} \frac{n!}{z_{\mu,n}} p_\mu^\#. \quad (46)$$

For $\lambda \vdash n$ recall that E_λ is the Young idempotent associated to λ .

Theorem 5.4. The isomorphism φ sends

$$\frac{1}{\dim L^\lambda} \boxed{E_\lambda} \xrightarrow{\varphi} s_\lambda^*.$$


Proof. Recall that

$$\left(\frac{1}{\dim L^\lambda}\right) E_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{n!} C_{\mu,n},$$

while

$$s_\lambda^* = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_{\mu,n}} p_\mu^\#.$$

The result then follows from (46). \square

The previous theorems gave graphical realizations of some important bases of Λ^* . Now we go the other way, and describe Khovanov's curl generators \tilde{c}_k and c_k as elements of Λ^* . It is this description that makes an explicit connection between \mathcal{H}' and the transition and co-transition measures of Kerov.

Theorem 5.5. The isomorphism φ sends:

1. $\tilde{c}_k \mapsto \hat{m}_k \in \Lambda^*$,
2. $c_k \mapsto p_1^\# \check{m}_k = \hat{b}_{k+2} \in \Lambda^*$.

Proof. Let $\lambda \vdash n$, then from Lemma 4.10 and Proposition 2.14 we have

$$[\varphi(\tilde{c}_k)](\lambda) = \tilde{\chi}^\lambda(\text{pr}_n(J_{n+1}^k)) = \hat{m}_k(\lambda)$$

and

$$[\varphi(c_k)](\lambda) = \tilde{\chi}^\lambda\left(\sum_{i=1}^n s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i\right) = p_1^\#(\lambda) \check{m}_k(\lambda) = \hat{b}_{k+2}(\lambda).$$

\square

Remark 5.6. In [5], Farahat and Higman used the inductive structure of symmetric groups to construct a \mathbb{C} -algebra known as the Farahat-Higman algebra $\mathcal{K}_{\mathbb{C}}$ (see also Example 24, Section I.7, [16]). It follows from, for example [8], that there is an algebra isomorphism $\mathcal{K}_{\mathbb{C}} \cong \Lambda^*$, and the functors $f_n^{\mathcal{H}'}$ can also be used to give a direct isomorphism between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and $\mathcal{K}_{\mathbb{C}}$. So in principle all of the appearances of shifted symmetric functions in the previous sections could be rephrased in the language of the Farahat-Higman algebra.

Remark 5.7. Theorem 5.5 and Remark 3.8 together imply that the recursive relationships for $\{\hat{m}_k\}$ and $\{\hat{b}_k\}$ in Remark 2.13 and $\{c_k\}$ and $\{\tilde{c}_k\}$ in Lemma 4.3 are both consequences of the well-known relationship between the elementary and homogeneous symmetric functions:

$$\sum_{i=0}^k (-1)^i e_i h_{n-i} = 0.$$

Example 5.8. In Λ^* we have $p_{(2)}^{\#} p_{(2)}^{\#} = p_{(2,2)}^{\#} + 4p_{(3)}^{\#} + 2p_{(1,1)}^{\#}$. In $\text{End}_{\mathcal{H}'}(\mathbb{1})$ the local relations can be used to compute the corresponding equation:

5.1 Involutions on $\text{End}_{\mathcal{H}'}(\mathbb{1})$

In [13], Khovanov introduced three involutive autoequivalences on \mathcal{H}' . Only one of these, which we denote as ξ , acts non-trivially on $\text{End}_{\mathcal{H}'}(\mathbb{1})$ where it gives an involutive algebra automorphism. For $D \in \text{Hom}_{\mathcal{H}'}(Q_{\epsilon_1}, Q_{\epsilon_2})$, we have

$$\xi(D) := (-1)^{c(D)} D$$

where $c(D)$ is the total number of dots and crossings in the diagram. Thus, in $\text{End}_{\mathcal{H}'}(\mathbb{1})$:

$$c_k \xrightarrow{\xi} (-1)^k c_k, \tag{47}$$

$$\tilde{c}_k \xrightarrow{\xi} (-1)^k \tilde{c}_k, \tag{48}$$

$$\alpha_k \xrightarrow{\xi} (-1)^{k-1} \alpha_k. \tag{49}$$

In Section 4 of [18], Okounkov and Olshanski identified an involutive algebra automorphism $I : \Lambda^* \rightarrow \Lambda^*$ which acts on $f \in \Lambda^*$ such that for $\lambda \in \mathcal{P}$,

$$[I(f)](\lambda) = f(\lambda'),$$

where λ' is the conjugate partition to λ . In particular

$$I(s_\lambda^*) = s_{\lambda'}^*, \quad (50)$$

$$I(e_k^*) = h_k^*, \quad (51)$$

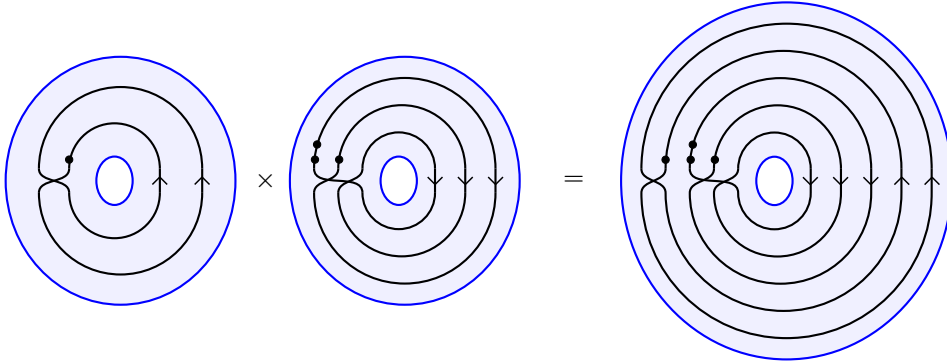
$$I(p_k^\#) = (-1)^{k-1} p_k^\#. \quad (52)$$

Proposition 5.9. The involution ξ on $\text{End}_{\mathcal{H}'}(\mathbb{1})$ coincides with the involution I on Λ^* .

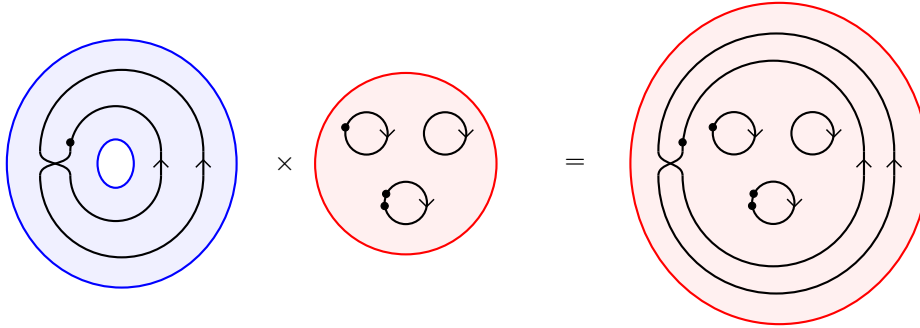
Proof. This follows from the fact that $\{\alpha_k\}_{k \geq 1}$, (respectively $\{p_k^\#\}_{k \geq 1}$) generate $\text{End}_{\mathcal{H}'}(\mathbb{1})$ (resp. Λ^*), $\varphi(\alpha_k) = p_k^\#$, and a comparison of (49) and (52). \square

5.2 A graphical construction of the action of $W_{1+\infty}$ on Λ^*

In [4], the trace $\text{Tr}(\mathcal{H}')$ (or zeroth Hochschild homology) of \mathcal{H}' is shown to be isomorphic as an algebra to a quotient of the W-algebra $W_{1+\infty}$. Like the center $\text{End}_{\mathcal{H}'}(\mathbb{1})$, which is the algebra of closed planar diagrams, the trace $\text{Tr}(\mathcal{H}')$ has a purely graphical description, as the space of annular diagrams modulo Khovanov's local diagrammatic relations. More precisely, the underlying vector space of $\text{Tr}(\mathcal{H}')$ is isomorphic to the span of annular diagrams, where an annular diagram \tilde{f} is by definition a diagram obtained by taking an endomorphism $f \in \text{End}_{\mathcal{H}'}(X)$ for some object $X \in \mathcal{H}'$, and closing it up to the right in an annulus. The multiplication in $\text{Tr}(\mathcal{H}')$ is given by gluing annuli around one another:



The action of $\text{Tr}(\mathcal{H}')$ on $\text{End}_{\mathcal{H}'}(\mathbb{1})$ then acquires a graphical description: given an annular diagram $\tilde{f} \in \text{Tr}(\mathcal{H}')$ and a closed planar diagram $f \in \text{End}_{\mathcal{H}'}(\mathbb{1})$, the closed planar diagram $\tilde{f}g \in \text{End}_{\mathcal{H}'}(\mathbb{1})$ is given by inserting a planar neighborhood of the closed diagram g into the middle of the annulus:



Thus, via the isomorphisms

$$\text{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*, \quad \text{Tr}(\mathcal{H}') \cong W_{1+\infty}$$

of Theorem 5.3 and [4], respectively, we obtain a purely graphical construction of the action of $W_{1+\infty}$ on Λ^* . Such an action was first considered by Lascoux-Thibon in [14].

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